Matrix Quantum Mechanics

$$Z_N(\kappa) = \int [DM(t)]\, e^{-N\int_{-T/2}^{T/2} dt\, \mathrm{tr}\left[\frac{1}{2}\dot{M}^2(t) + \frac{1}{2}M^2(t) - \frac{\kappa}{3}M^3(t)\right]}$$

N×N Hermitian matrix

$$\left(N = \frac{1}{\hbar} \right)$$

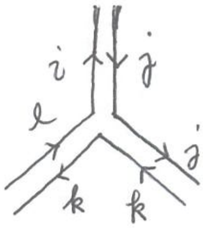
§ perturbative expansion (Feynmann diagram)

(i) free propagator

$$\langle M_i^{\ j}(t_1)\, M_k^{\ l}(t_2) \rangle_{\kappa=0} = \frac{1}{N}\, \delta_i^{\ l}\, \delta^j_{\ k} \cdot \underline{\frac{e^{-|t_1-t_2|}}{2}}$$

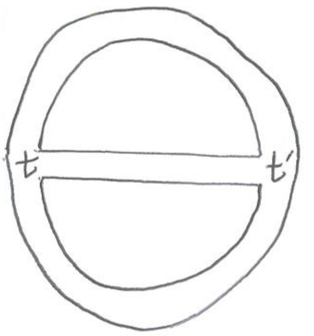
(ii) Wick's contraction (same as before)

(iii) vertex



$$\underline{\left(-N\kappa/3\right)\int_{-T/2}^{T/2} dt}\,.$$

e.g. a connected vacuum diagram



$$\sim \underbrace{(-N\kappa/3)^2 \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'}_{\text{vertices}} \underbrace{\left(\frac{1}{N}\right)^3 \left(\frac{e^{-|t-t'|}}{2}\right)^3}_{\text{propagator}} \times$$

$$\times \underbrace{N^3}_{\text{loops = faces}} \times$$

thus, one can express the connected vacuum diagrams as

$$\lim_{T\to\infty} \log\left[Z_N(\kappa)/Z_N(0)\right] = \sum_g N^{2-2g} \sum_{\tilde{L}} \kappa^V \int_{-\infty}^{\infty} \prod_{i=1}^{V} dt_i \underbrace{\prod_{\langle i,j\rangle}}_{\text{propagators}} e^{-|t_i-t_j|}$$

dual lattice
of the discretized
random surface
(feynman graphs)
of given topology

⇐ this is precisely of the same form which arises in the two dimensional quantum gravity with matter
a scalar.

remark notice that the integral below

$$\int_{-T/2}^{T/2} \prod_{i=1}^{V} dt_i \prod_{\langle i,j \rangle} e^{-|t_i - t_j|} \propto T ,$$

due to the translational sym. in "t" direction.

§ free fermions.

① note that the Euclidean path-integral computes the amplitude below.

$$Z_N(k) = \int_{\substack{M(t=-T/2)=M^i\\ M(t=+T/2)=M^f}} [DM(t)]\,(\cdots) \qquad = \langle \text{final} \,|\, e^{-\overset{N}{\vee}HT} \,|\, \text{initial} \rangle$$

Thus, one can argue that

$$\lim_{T\to\infty} \cancel{[}\, \log(\underset{\propto T}{Z_N(k)})/T \,\cancel{]} = -NE_{gr} \nearrow \text{ground state energy}$$

② Hamiltonian.

(i) "something similar to the Cartesian coordinate."

$$\mathcal{L} = \frac{1}{2}\dot{M}_{ij}\dot{M}_{ji} + U(M)$$

$$\hookrightarrow \quad H = -\hbar^2 \frac{\partial^2}{\partial M_{ij}^2} + U(M) \qquad \hbar \sim \frac{1}{N}$$

(ii) "polar" coordinate

$$M(t) = U^\dagger(t)\,\underset{\substack{\uparrow\\ \text{radial}}}{\Lambda(t)}\,\overset{\text{angular}}{U(t)}$$

$U(t) \in SU(N)$

$\Lambda(t) = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$

One can then show that

$$\mathrm{tr}\left[\frac{1}{2}\dot{M}^2(t)\right] = \underbrace{\mathrm{tr}\left[\frac{1}{2}\dot{\Lambda}^2(t)\right]}_{=\sum_{i=1}^{N}\frac{1}{2}\dot{\lambda}_i^2(t)} + \underbrace{\mathrm{tr}\left(\left[\Lambda(t), \dot{U}(t)U^\dagger(t)\right]^2\right)}_{(a)}$$

(a) one can decompose $\dot{U}(t)U^\dagger(t)$ in terms of $SU(N)$ generators,

$$\dot{U}(t)U^\dagger(t) = \frac{1}{\sqrt{2}}\sum_{i<j}\left(i\dot{\alpha}_{ij}(t)\underset{\text{sym.}}{T_{(ij)}} + \dot{\beta}_{ij}(t)\underset{\text{anti-sym.}}{T'_{[ij]}}\right) + \underbrace{i\sum_{i=1}^{N-1}\dot{\gamma}_i(t)H_i}_{\text{Cartan Subalgebra}}$$

e.g.

$$T_{12} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline 0 & & 0 \end{array}\right)$$

e.g.

$$T'_{12} = \left(\begin{array}{cc|c} 0 & i & 0 \\ -i & 0 & \\ \hline 0 & & 0 \end{array}\right)$$

(※ Both $T_{(ij)}$ & $T'_{[ij]}$ are Hermitian.)

$\underset{\sim}{(b)}$ the one can rewrite the 2nd term as

$$\operatorname{tr}\left([\Lambda(t),\ \dot{U}(t)U^{\dagger}(t)] \right)$$

$$= \frac{1}{2} \sum_{i<j}' (\lambda_i - \lambda_j)^2 \left(\dot{\alpha}^2_{ij}(t) + \dot{\beta}^2_{ij}(t) \right)$$

※ notice that there is no $\dot{\gamma}_i(t)$

dependence on

$$\Leftrightarrow \mathcal{L} = \sum_{i=1}^{N} \frac{1}{2} \dot{\lambda}_i(t) + \sum_{i<j}' \frac{1}{2} (\lambda_i - \lambda_j)^2 \left(\dot{\alpha}^2_{ij}(t) + \dot{\beta}^2_{ij}(t) \right)$$

(when $\mathcal{L} = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$, then the Hamiltonian can be expressed as $H = -\hbar^2 \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \left(g^{ab} \sqrt{g} \frac{\partial}{\partial x^b} \right)$)

thus, the Hamiltonian of our interest is

$$\boxed{\begin{aligned} H = -\frac{\hbar^2}{2} \Bigg[& \sum_i \frac{1}{\Delta^2(\lambda)} \frac{\partial}{\partial \lambda^i} \left(\Delta^2(\lambda) \frac{\partial}{\partial \lambda^i} \right) + \sum_{i<j}' \frac{1}{(\lambda_i - \lambda_j)^2} \underbrace{\left(\Pi^{2(\alpha)}_{ij} + \Pi^{2(\beta)}_{ij} \right)}_{\text{" angular mom."}} \Bigg] \\ & + \sum_i \left(\frac{1}{2} \lambda_i^2 - \frac{k}{3!} \lambda_i^3 \right) \end{aligned}}$$

note also that

$$\frac{1}{\Delta^2(\lambda)}\frac{\partial}{\partial\lambda^i}\left(\Delta^2(\lambda)\frac{\partial}{\partial\lambda^i}\quad\right) = \frac{1}{\Delta(\lambda)}\frac{\partial^2}{\partial\lambda^{i\,2}}\Delta(\lambda)$$

③ ground state wavefunction & energy

(i) note that the system has a <u>global</u> SU(N) sym.

→ each energy eigenstate has to be in a

$\underset{\mathbf{r}}{\underline{\text{rep.}}}$ of SU(N) $\qquad \Psi_{\mathbf{r}}(\lambda_i, \alpha_{ij}, \beta_{ij})$

(※ to illustrate this, let's consider a system with SO(3) rotational symmetry, energy eigenstate can be expressed as, e.g., $\psi(r,\theta,\phi) = f_{lm}(r)\underbrace{Y_l^m(\theta,\phi)}_{\text{spherical harmonics}}$)

(ii) no $\dot{\gamma}_i^2(t)$ term in the Lagrangian $\mathcal{L}$.

$\Rightarrow \quad \underbrace{\Pi_i^{\gamma}}\,\Psi_{\mathbf{r}}(\lambda_i, \alpha_{ij}, \beta_{ij}) = 0.$ $\qquad$ (, similar to $L^3|l\,m\rangle = 0$)

only those rep's r of $SU(N)$ that have a state with all weights equal to zero are allowed in the ~~quantum mechanical system of our interest.~~ matrix QM.

(iii) ground state must be a <u>singlet</u> under $SU(N)$ 是 which carries no angular momentum!

$$\left(\Pi^{\alpha}_{ij} = \Pi^{\beta}_{ij} = 0 \right)$$

$L^{\pm}|l\,m\rangle = 0$

similar to the constraint

singlet under $SU(N)$

$$\Psi_{gr} = \Psi_{gr}(\lambda_i)$$

- a. no angle dependence
- b. <u>symmetric</u> in the exchange of $\lambda_i \leftrightarrow \lambda_j$

(∵ Weyl group) e.g. $\underset{SU(N)}{\underset{\cap}{U}} = i\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$

acts trivially on Ψ_{gr}

$$\Leftrightarrow \Psi_{gr}(\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_N)$$

$$= \Psi_{gr}(\lambda_2, \lambda_1, \lambda_3, \cdots \lambda_N)$$

$$H \Psi_{gr}(\lambda_i) = E_{gr} \Psi_{gr}(\lambda_i)$$

(iv) fermions

(non-interacting)

anti-symmetric wave function !!

$$\chi_{gr}(\lambda_i) \equiv \underset{\text{anti-sym.}}{\underset{\downarrow}{\Delta(\lambda)}} \, \underset{\text{symmetric}}{\underset{\downarrow}{\Psi_{gr}(\lambda_i)}}$$

then one can argue that

$$H' \chi_{gr}(\lambda_i) = E_{gr} \chi_{gr}(\lambda_i)$$

where

$$\sum_i$$

$$H' = \sum_{i=1}^{N} \left\{ -\frac{\hbar^2}{2} \frac{d^2}{d\lambda_i^2} + \underbrace{\frac{1}{2}\lambda_i^2 - \frac{\kappa}{3!}\lambda_i^3}_{= V(\lambda_i)} \right\}$$

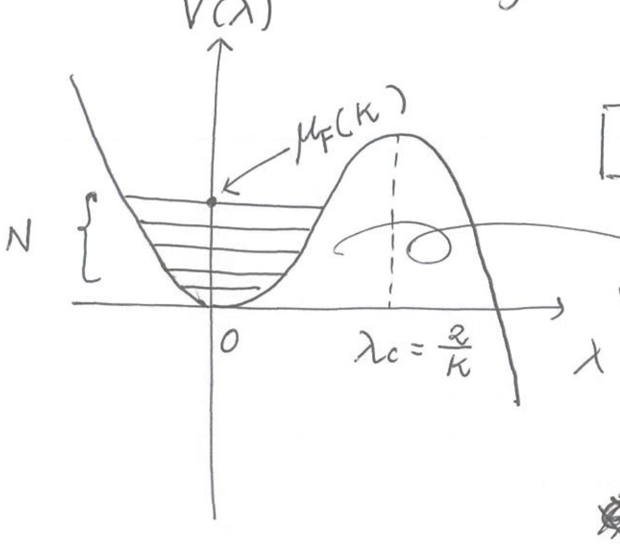
= sum of single particle non-relativistic Hamiltonian

In other words, the problem has been reduced to the physics of $\textcircled{N}$ non-relativistic identical fermions in the potential $V(\lambda) = \frac{1}{2}\lambda^2 - \frac{\kappa}{3!}\lambda^3$

(V) ground state energy

$$E_{gr} = \sum_{i=1}^{N} \varepsilon_i \quad \cdots \text{“fermi level } \mu_F\text{”} !!$$

the lowest N energy levels of $H_{single} = -\frac{\hbar^2}{2}\frac{d^2}{d\lambda^2} + V(\lambda)$



$\boxed{\text{remark}}$ In the limit $N \to \infty$ ($\hbar \to 0$),

① those eigenstates are semi-stable whose decay time is exponentially long! (reliable)

② energy spacing → 0.

a.

In the $N \to \infty$ limit (semi-classical limit)

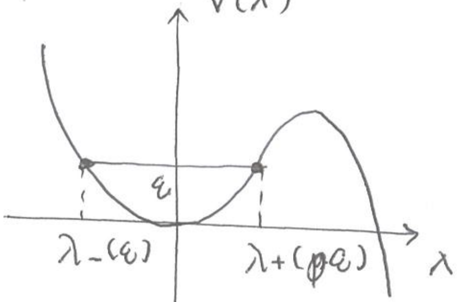
one can make use of the WKB method (

Bohr-Sommerfeld quantization condition)

to obtain ε_n $n=1,2,\cdots,N$

$$\overset{\sqrt{2(\varepsilon_n - V(\lambda))}}{\oint d\lambda\, P_n(\lambda)} = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

$$\cong 2\pi\frac{n}{N}$$



$$\leftrightarrow \int_{\lambda_-(\varepsilon_n,k)}^{\lambda_+(\varepsilon_n,k)} d\lambda\, \sqrt{2(\varepsilon_n - V(\lambda))} = \pi\frac{n}{N}$$

turning points ~~$\sqrt{2(\varepsilon_F - V(\lambda))}$~~

~~continuum limit $\frac{n}{N} = x$ ∴ $\varepsilon_n \cong \varepsilon(x=\frac{n}{N})$~~

※ fermi level ε_F

$$\boxed{\int_{\lambda_-(\varepsilon_F)}^{\lambda_+(\varepsilon_F)} d\lambda\, \sqrt{2(\varepsilon_F - V(\lambda))} = \pi}$$

b. continuum limit

$$\frac{n}{N} = x, \qquad \varepsilon_n = \varepsilon\left(x = \frac{n}{N}\right)$$

then, the quantization condition becomes

$$\frac{1}{\pi}\int_{\lambda_-(\varepsilon)}^{\lambda_+(\varepsilon)} \sqrt{2(\varepsilon - V(\lambda))} = \underbrace{x(\varepsilon)}$$

monotonic function of ε

$dx = \underset{\substack{\uparrow \\ \text{energy density}}}{\rho(\varepsilon)}\, d\varepsilon$

How to obtain $\rho(\varepsilon)$?

$$\frac{1}{\pi}\int_{\lambda_-(\varepsilon-\delta\varepsilon)}^{\lambda_+(\varepsilon+\delta\varepsilon)} \sqrt{2(\varepsilon+\delta\varepsilon - V(\lambda))} = \underset{\substack{\downarrow \\ \text{lineer in } \delta\varepsilon}}{\delta x} + x$$

$$\Leftrightarrow \cancel{\frac{1}{\pi}\sqrt{2(\varepsilon - V(\lambda))}\Big|_{\lambda=\lambda_-(\varepsilon)}^{\lambda=\lambda_+(\varepsilon)}} \to 0$$

$$\lambda_\pm(\varepsilon+\delta\varepsilon) = \lambda_\pm(\varepsilon) + \delta\lambda_\pm \quad \to \text{linear in } \delta\varepsilon$$

LHS

$$\frac{1}{\pi}\cancel{\sqrt{2(\varepsilon - V(\lambda_+(\varepsilon)))}}\cdot\delta\lambda_+ - \frac{1}{\pi}\cancel{\sqrt{2(\varepsilon - V(\lambda_-(\varepsilon)))}}\cdot\delta\lambda_- \qquad (\to 0)$$

$\because$ $\lambda_\pm(\varepsilon)$ are turning points.

$$+ \frac{1}{\pi}\int_{\lambda_-(\varepsilon)}^{\lambda_+(\varepsilon)} d\lambda \frac{1}{\sqrt{2(\varepsilon - V(\lambda))}} \cdot \delta\varepsilon$$

RHS

δx.

this implies that

$$\rho(\varepsilon) = \frac{1}{\pi} \int_{\lambda_-(\varepsilon)}^{\lambda_+(\varepsilon)} d\lambda \, \frac{1}{\sqrt{2(\varepsilon - V(\lambda))}}$$

//

thus, one can express the ground-state energy of our interest E_{gr} as follows

$$E_{gr} = N \int_0^1 dx \; \varepsilon(x)$$

$$= N \int_0^{\varepsilon_F} d\varepsilon \; \rho(\varepsilon) \cdot \varepsilon$$

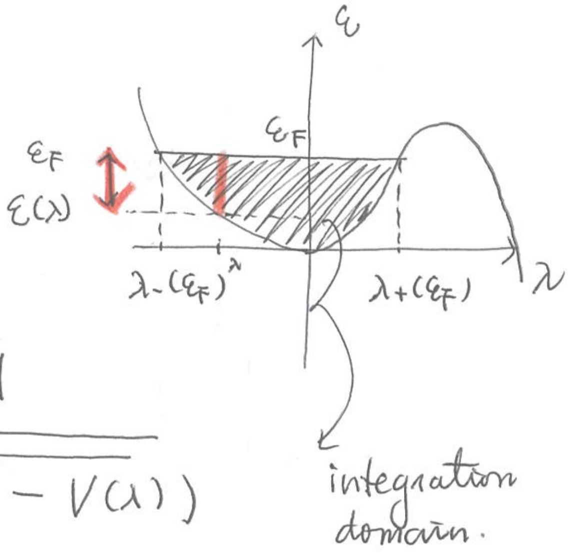
i.e., $\lim\limits_{T \to \infty} \left(- \log Z_N(k) / T \right) = N E_{gr}(k)$

$$= N^2 \int_0^{\varepsilon_F(k)} d\varepsilon \; \rho(\varepsilon) \cdot \varepsilon$$

//

(before proceeding further) <u>Sanity check</u>

$$1 \overset{?}{=} \int_0^{\varepsilon_F} d\varepsilon \; \rho(\varepsilon)$$



RHS :

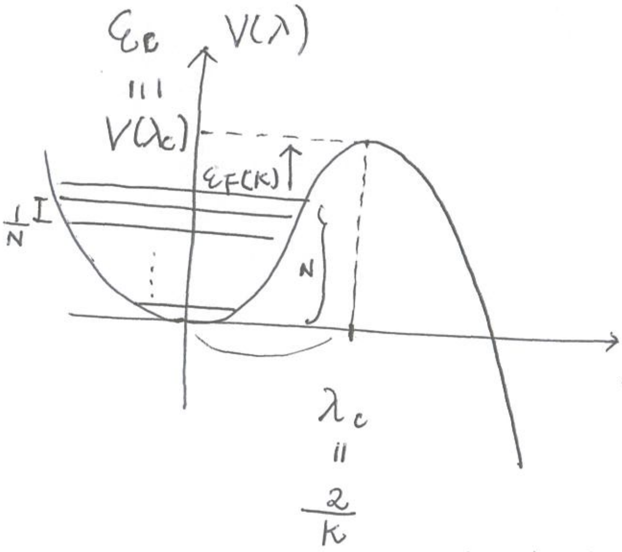
$$\frac{1}{\pi} \int_0^{\varepsilon_F} d\varepsilon \int_{\lambda_-(\varepsilon)}^{\lambda_+(\varepsilon)} d\lambda \; \frac{1}{\sqrt{2(\varepsilon - V(\lambda))}}$$

$$= \frac{1}{\pi} \int_{\lambda_-(\varepsilon_F)}^{\lambda_+(\varepsilon_F)} d\lambda \int_{\varepsilon(\lambda)}^{\varepsilon_F} d\varepsilon \; \frac{1}{\sqrt{2(\varepsilon - V(\lambda))}}$$

$$= \sqrt{2(\varepsilon_F - V(\lambda))}$$

$= 1$ due to the def. of the Fermi level

§ Double Scaling Limit

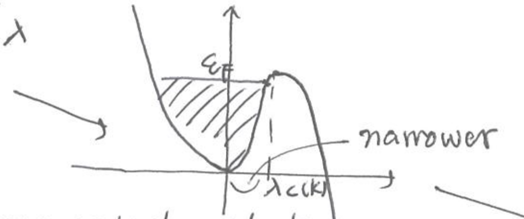


$$V(\lambda) = \frac{1}{2}\lambda^2 - \frac{k}{3!}\lambda^3$$

$$V'(\lambda_c) = \lambda_c - \frac{k}{2}\lambda_c^2 = 0$$

$$\mathcal{E}_c \equiv V(\lambda_c) = \frac{1}{2}\left(\frac{2}{k}\right)^2 - \frac{k}{6}\left(\frac{2}{k}\right)$$

$$= \underbrace{\left(2 - \frac{4}{3}\right)}_{\frac{2}{3}} \frac{1}{k^2}$$



as you increase k towards k_c

"increasing function of k."

something "singular" happens when $\mathcal{E}_F^{(k)} \to \mathcal{E}_c$,

which is only feasible when $k \to k_c$ as $N \to \infty$

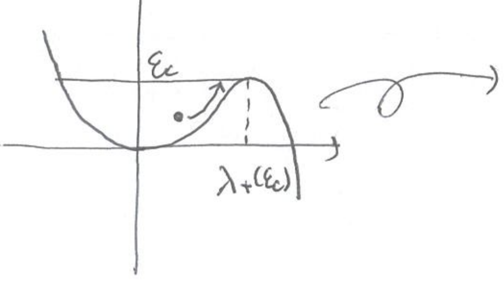
$$\left(\text{※ note that } \frac{1}{\pi}\int_{\lambda_-(\mathcal{E}_F)}^{\lambda_+(\mathcal{E}_F)} d\lambda \sqrt{2(\mathcal{E}_F(k) - V(\lambda, k))} \overset{\text{fixed.}}{= 1} \right)$$

① density $\rho(\mathcal{E}_F)$ becomes singular when $\mathcal{E}_F \to \mathcal{E}_c$

why?

$$\rho(\mathcal{E}_c) = \underbrace{\int_{\lambda_-(\mathcal{E}_c)}^{\lambda_+(\mathcal{E}_c)} d\lambda \frac{1}{\sqrt{2(\mathcal{E}_c - V(\lambda))}}} \to \infty$$

why?



the classical trajectory with $\mathcal{E} = \mathcal{E}_c$ spends $\underbrace{\text{infinitely}}$ long time near $\lambda \sim \lambda_+(\mathcal{E}_c)$

(more precisely, logarithmic diverging)

? logarithmic divergence ?

$$\varepsilon_F = \varepsilon_c - \underset{\nwarrow\ \mu \ll 1}{\mu}$$

$$\rho(\varepsilon_F) \cong \frac{1}{\pi}\int_{\lambda_-(\varepsilon_F)}^{\lambda_+(\varepsilon_F)} d\lambda \frac{1}{\sqrt{2(\underbrace{\varepsilon_c - \mu - \underbrace{V(\lambda_c)}_{= \varepsilon_c}}_{= -\mu} + \frac{1}{2}\underbrace{|V''(\lambda_c)|}_{= -1}(\lambda-\lambda_c)^2)}}$$

focus on the region near λ_+ or λ_c

$$\cong -\log\left[\underbrace{(\lambda_c - \lambda_+)}_{\text{linear in } \mu} + \sqrt{\underbrace{(\lambda_c-\lambda_+)^2}_{\text{quadratic in } \mu} - 2\mu}\,\right] \quad \text{up to sign}$$

$$\sim \underbrace{-\log \mu}_{\text{logarithmic divergence!}} \ !$$

②
$$NE_{gr} = N^2 \int_0^{\varepsilon_F = \varepsilon_c - \mu} d\varepsilon\ \varepsilon\, \rho(\varepsilon)$$

$$\sim -N^2\mu^2 \log \mu + (\text{sub-leading terms})$$

double scaling limit: $\boxed{N\mu = \text{fixed}}$ as $\mu \to 0$, $N \to \infty$

③ $\varepsilon_F = \varepsilon_F(\kappa)$

when κ is near κ_c, i.e. $\kappa = \kappa_c - k$

$$\varepsilon_F = \varepsilon_c - \mu(k)$$

second-quantized fermion theory.

c=1 string field theory

$\boxed{Q}$ $\mu = \mu(k)$ ↜ ~~functi~~ determine the expression.

④ all-genus expansion. (use the heat-kernel method)

$$\frac{\partial \rho}{\partial \mu} = N \cdot \frac{1}{\pi} \cdot \mathrm{Im} \int_0^\infty dT \, e^{-iN\mu T} \frac{T/2}{\sinh(T/2)}$$

.... exact in terms of $(N\mu)$

large $(N\mu)$ - expansion.
but fixed value

$$\rho(\mu) = \frac{1}{\pi}\left\{ -\log\mu + \sum_{m=1}^{\infty} \left(2^{2m-1}-1\right) \frac{|B_{2m}|}{m} \underbrace{(2N\mu)^{-2m}}_{\text{corrections.}} \right\}$$

⑤ Punctures ← vertex op.

③ when κ is near κ_c, i.e, $\kappa = \kappa_c - k$, then $\varepsilon_F = \varepsilon_c - \mu(k)$

$\boxed{Q}$: How to determine $\mu(k)$

(i) $$\int_{\lambda_-(\varepsilon_F)}^{\lambda_+(\varepsilon_F)} d\lambda \sqrt{2(\varepsilon_F - \underbrace{V(\lambda)})} = \pi$$

$$\frac{1}{\kappa} d(\kappa\lambda)$$

$$\frac{1}{2}\lambda^2 - \frac{\kappa}{3!}\lambda^3 = \left[\frac{1}{2}(\kappa\lambda)^2 - \frac{1}{3!}(\kappa\lambda)^3\right]\kappa^{-2}$$

$$\Rightarrow \int_{\lambda'_+(\underset{\kappa^{+2}}{\varepsilon_F})}^{\lambda'_-(\overset{\kappa^{+2}}{\varepsilon_F})} d\lambda' \sqrt{2(\kappa^2\varepsilon_F - \underbrace{V'(\lambda')})} = \pi\kappa^2$$

$$= \frac{1}{2}\lambda'^2 - \frac{1}{3!}\lambda'^3$$

<u>turning points</u>

$$\varepsilon_F = \frac{1}{2}(\lambda_\pm(\varepsilon_F))^2 - \frac{\kappa}{3!}(\lambda_\pm(\varepsilon_F$$

$$\kappa^2\varepsilon_F = \frac{1}{2}(\underbrace{\kappa\lambda_\pm(\varepsilon_F)})^2 - \frac{1}{3!}(\underbrace{\kappa\lambda_\pm(\varepsilon_F)})$$

$$= \lambda'_\pm(\kappa^2\varepsilon_F)$$

thus, one can argue that

$$\delta\kappa^2 = \frac{1}{\pi}\,\delta(\kappa^2\varepsilon_F)\cdot \underbrace{\int_{\lambda'_-(\kappa^2\varepsilon_F)}^{\lambda'_+(\kappa^2\varepsilon_F)} d\lambda' \frac{1}{\sqrt{2(\kappa^2\varepsilon_F - V'(\lambda'))}}}_{}$$

$$= \rho(\varepsilon_F)$$

$$\therefore \quad \frac{\delta\kappa^2}{\delta(\kappa^2\varepsilon_F)} = \rho(\varepsilon_F)$$

④ All-Genus Expansion.

energy eigenstates

$$\sum_n \frac{1}{\varepsilon_n - z} |n\rangle\langle n|$$

(i) <u>define</u>

$$f(z) = \mathrm{Tr}\left[\frac{1}{H-z}\right]$$

singular when $z = \varepsilon_n$ for some n.

then,

$$f(z+i\varepsilon) = \sum_n \frac{1}{\varepsilon_n - z - i\varepsilon} = \int_{\mathbb{R}} d\varepsilon' \underbrace{\left(\sum_n \delta(\varepsilon'-\varepsilon_n)\right)}_{\tilde{\rho}(\varepsilon')\text{ normalization not yet!}} \cdot \frac{1}{\varepsilon' - z - i\varepsilon}$$

$\mathbb{R}$ ← real line

$$= \fint d\varepsilon' \, \tilde{\rho}(\varepsilon') \frac{1}{\varepsilon' - z} + i\pi \tilde{\rho}(z).$$

$$\therefore \quad \tilde{\rho}(z) = \frac{1}{\pi} \mathrm{Im}\left\{\mathrm{Tr}\left[\frac{1}{H-z-i\varepsilon}\right]\right\} \quad \cdots ⊛$$

set: $z = \varepsilon_F$

note that the Hamiltonian of our interest is

$$H_{\text{single}} = -\frac{1}{2N^2}\frac{\partial^2}{\partial\lambda^2} + V(\lambda)$$

$V(\lambda) = \frac{1}{2}\lambda^2 - \frac{\kappa}{3!}\lambda^3$

(ii) double-scaling limit

~~$\varepsilon_F = \varepsilon_c - N^{-1}\mu$~~

~~$\lambda = \lambda_c - N^{-1/2}y$~~

~~as $N \to \infty \left(\approx \frac{1}{\hbar}\right)$.~~

$$\frac{1}{N} = \hbar \alpha$$

$$\varepsilon_F = \varepsilon_c - \alpha\mu \qquad \text{as} \quad \alpha \to 0$$

$$\lambda = \lambda_c - N^{\cancel{1/2} \to -1/2} y$$

$$H_{single} - \varepsilon_F \simeq -\frac{1}{2N^2}\left(N\frac{\partial^2}{\partial y^2}\right) + \underbrace{(V(\lambda_c) - \varepsilon_F)}_{= +\frac{1}{N}(\hbar^{-1}\mu)} + \frac{1}{2}\overset{"-1"}{\cancel{V''(\lambda_c)}}\underbrace{(\lambda-\lambda_c)^2}_{= N^{-1}y^2}$$

$$= N^{-1}\left[-\frac{1}{2}\frac{\partial^2}{\partial y^2} - \frac{1}{2}y^2 + (\hbar^{-1}\mu) - i\varepsilon\right]$$

thus, the un-normalized density of states ⊛ becomes

$$\rho(\varepsilon_F) \cong \frac{N}{\pi}\, \mathrm{Im}\left\{\mathrm{Tr}\left[\frac{1}{-\frac{1}{2}\frac{\partial^2}{\partial y^2} \underset{\uparrow}{-\frac{1}{2}y^2} + (\hbar^{-1}\mu) - i\varepsilon}\right]\right\}$$

inverted harmonic oscillator

in the <u>double-scaling limit</u>

(iii) inverted harmonic oscillator ~ harmonic oscillator + analytic continuation

Consider the transition amplitude below

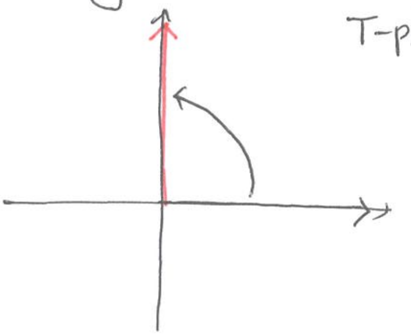
$$\langle y_f | \frac{1}{-\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\omega^2 y^2 + \hbar^{-1}\mu - i\varepsilon} | y_i \rangle$$

$$\langle y_f | \frac{1}{-\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\omega^2 y^2 + (\hbar^{-1}\mu) - i\varepsilon} | y_i \rangle$$

$$= \int_0^\infty dT \; e^{-\hbar^{-1}\mu T} \boxed{e^{+i\varepsilon T}} \int_{\substack{y(0)=y_i \\ y(T)=y_f}} \mathcal{D}y(t) \; e^{-\int_0^T dt \left[-\frac{1}{2}\dot{y}^2(t) + \frac{1}{2}\omega^2 y^2(t) \right]}$$

$$= \sqrt{\frac{\omega}{2\pi \sinh \omega T}} \; e^{-\omega\left[(y_f^2 + y_i^2)\cosh \omega T - 2y_f y_i\right]/2\sinh \omega T}$$

analytic continuation



$$\int_0^\infty dT \rightarrow \int_0^{i\infty} dT$$

~~$= +i \int_0^\infty dT' \; e^{-i\mu T'} \underbrace{e^{-\varepsilon T'}}_{\text{convergent factor}} \sqrt{\frac{-i}{2\pi \sinh T'}} \; e^{+i\left[(y_f^2 + y_i^2)\cosh T' - 2y_f y_i\right]/2\sinh T'}$~~

~~another analytic continuation~~

~~$\omega = -i$ so that~~

~~$\omega T = T'$~~

thus

$$\mathrm{Tr}\left[\frac{1}{-\frac{1}{2}\frac{\partial^2}{\partial y^2}-\frac{1}{2}y^2+(\hbar^{-1}\mu)-i\varepsilon}\right]$$

analytic continuation
$\omega = -i$

$$= i\int_0^\infty dT'\, \underbrace{e^{-i(\hbar^{-1}\mu)T'}}_{\int_{-\infty}^{\infty} dy}\, e^{-\varepsilon T'}\sqrt{\frac{-i}{2\pi\sinh T'}}\cdot e^{\cancel{2}i y^2 \boxed{(\cosh T'-1)/\sinh T'}}$$

$$= \frac{\sinh T'/2}{\cosh T'/2}$$

this integral is well-defined?

$$= + i\int_0^\infty dT'\, e^{-i(\hbar^{-1}\mu)T'}\, e^{-\varepsilon T'}\,\frac{1}{2\sinh T'/2}$$

perform the integral over y first

Although the above integral is divergent due to the singular point at $T'=0$, one can argue that the divergent behavior is independent of $(\hbar^{-1}\mu)$.

To see this, consider the derivative of ϱ w.r.t $(\hbar^{-1}\mu)$

$$\frac{\partial}{\partial(\hbar^{-1}\mu)}\tilde{\varrho}(\mu,\hbar) = \frac{N}{\pi}\,\mathrm{Im}\left\{\underbrace{\int_0^\infty dT'\, e^{-i(\hbar^{-1}\mu)T'}\,\frac{T'/2}{\sinh T'/2}}_{\text{!! convergent !!}}\right\}$$

remark : this integral rep. is of the ~~B~~ integral over Borel-plane

$$= \frac{N}{\pi}\,\mathrm{Im}\left[\psi^{(0)}\left(\frac{1}{2}+i(\hbar^{-1}\mu)\right)\right]$$

digamma function $\psi^{(0)}(z) = \frac{d}{dz}\log\Gamma(z)$

perturbative expansion

$$\tilde{\varrho}(\mu,\hbar) = \frac{N}{\pi}\left\{\underbrace{-\log\mu}_{} + \sum_{m=1}^{\infty}(2^{2m-1}-1)\overset{\text{Bernoulli number}}{|B_{2m}|}\frac{1}{m}\cdot(2\hbar^{-1}\mu)^{-2m}\right\}$$

one can see that $\tilde{\varrho}(\mu) = N\underbrace{\varrho(\mu)}_{\text{normalized density of states}}$

non-perturbative expansion

the above series is non-Borel summable

← non-perturbative terms are required (D-branes !) $\boxed{\text{Shenker}}$

more to be discussed ...

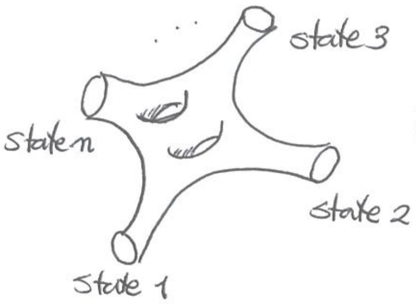
l_i-gon boundary

e.g. (i) Correlators

will corresponds to n punctures on the Riemann surface in the correspondence

$$\langle \mathrm{tr}\, M^{l_1}(t_1)\ \mathrm{tr}\, M^{l_2}(t_2) \cdots \mathrm{tr}\, M^{l_n}(t_n) \rangle$$

which computes



in QG.

However, $\{l_1, \dots, l_n\}$ can not be identified as the quantum number that specifies the quantum states on the boundary.

replace $\mathrm{tr}\, M^{l_1}(t_1) \rightarrow \mathrm{tr}\left[e^{-p_i M(t_i)}\right]$ ....

(ii) Liouville theory.

and so on.